UNPUBLISHED PRELIMINARY DATA

MH MPG Report 1541-TR 16

1 January 1964

Honeywell

N64-16759*

CODE-1

CR-53115

17p.

ROCKET BOOSTER CONTROL

SECTION 16

A MINIMAX CONTROL FOR
A PLANT SUBJECTED TO A
KNOWN LOAD DISTURBANCE

NASA Contract NASw-563

OTS PRICE

XEROX $ 1.60 ph

MICROFILM $ 0.80 mf.

MILITARY PRODUCTS GROUP RESEARCH DEPARTMENT

(NASA CR-53115;

MH-MPG-~~Report~~ 1541-TR-16) OTS: $1.60 ph, $0.80 mf 1 Jan~~uary~~ 1964 17p refs

ROCKET BOOSTER CONTROL.

SECTION 16:

A MINIMAX CONTROL FOR A PLANT SUBJECTED TO A KNOWN LOAD DISTURBANCE

(NASA Contract NASw-563)

☒ OTS

2

Prepared by: H. E. Gollwitzer
H. E. Gollwitzer
Assoc. Research Scientist

Supervised by: C. R. Stone
C. R. Stone
Research Supervisor

Approved by: O. H. Schuck
O. H. Schuck
Director
MPG Research

HONEYWELL
MILITARY PRODUCTS GROUP RESEARCH DEPARTMENT
Minneapolis, Minnesota

FOREWORD

This document is one of sixteen sections that comprise the final report prepared by the Minneapolis-Honeywell Regulator Company for the National Aeronautics and Space Administration under contract NASw-563. The report is issued in the following sixteen sections to facilitate updating as progress warrants:

1541-TR 1	Summary
1541-TR 2	Control of Plants Whose Representation Contains Derivatives of the Control Variable
1541-TR 3	Modes of Finite Response Time Control
1541-TR 4	A Sufficient Condition in Optimal Control
1541-TR 5	Time Optimal Control of Linear Recurrence Systems
1541-TR 6	Time-Optimal Bounded Phase Coordinate Control of Linear Recurrence Systems
1541-TR 7	Penalty Functions and Bounded Phase Coordinate Control
1541-TR 8	Linear Programming and Bounded Phase Coordinate Control
1541-TR 9	Time Optimal Control with Amplitude and Rate Limited Controls
1541-TR 10	A Concise Formulation of a Bounded Phase Coordinate Control Problem as a Problem in the Calculus of Variations
1541-TR 11	A Note on System Truncation
1541-TR 12	State Determination for a Flexible Vehicle Without a Mode Shape Requirement
1541-TR 13	An Application of the Quadratic Penalty Function Criterion to the Determination of a Linear Control for a Flexible Vehicle
1541-TR 14	Minimum Disturbance Effects Control of Linear Systems with Linear Controllers
1541-TR 15	An Alternate Derivation and Interpretation of the Drift-Minimum Principle
1541-TR 16	A Minimax Control for a Plant Subjected to a Known Load Disturbance

Section 1 (1541-TR 1) provides the motivation for the study efforts and objectively discusses the significance of the results obtained. The results of inconclusive and/or unsuccessful investigations are presented. Linear programming is reviewed in detail adequate for sections 6, 8, and 16.

It is shown in section 2 that the purely formal procedure for synthesizing an optimum bang-bang controller for a plant whose representation contains derivatives of the control variable yields a correct result.

In section 3 it is shown that the problem of controlling m components $(1 < m \leq n)$, of the state vector for an n-th order linear constant coefficient plant, to zero in finite time can be reformulated as a problem of controlling a single component.

Section 4 shows Pontriagin's Maximum Principle is often a sufficient condition for optimal control of linear plants.

Section 5 develops an algorithm for computing the time optimal control functions for plants represented by linear recurrence equations. Steering may be to convex target sets defined by quadratic forms.

In section 6 it is shown that linear inequality phase constraints can be transformed into similar constraints on the control variables. Methods for finding controls are discussed.

Existence of and approximations to optimal bounded phase coordinate controls by use of penalty functions are discussed in section 7.

In section 8 a maximum principle is proven for time-optimal control with bounded phase constraints. An existence theorem is proven. The problem solution is reduced to linear programming.

A backing-out-of-the-origin procedure for obtaining trajectories for time-optimal control with amplitude and rate limited control variables is presented in section 9.

Section 10 presents a reformulation of a time-optimal bounded phase coordinate problem into a standard calculus of variations problem.

A mathematical method for assessing the approximation of a system by a lower order representation is presented in section 11.

Section 12 presents a method for determination of the state of a flexible vehicle that does not require mode shape information.

The quadratic penalty function criterion is applied in section 13 to develop a linear control law for a flexible rocket booster.

In section 14 a method for feedback control synthesis for minimum load disturbance effects is derived. Examples are presented.

Section 15 shows that a linear fixed gain controller for a linear constant coefficient plant may yield a certain type of invariance to disturbances. Conditions for obtaining such invariance are derived using the concept of complete controllability. The drift minimum condition is obtained as a specific example.

In section 16 linear programming is used to determine a control function that minimizes the effects of a known load disturbance.

TABLE OF CONTENTS

ILLUSTRATIONS

A MINIMAX CONTROL FOR A PLANT SUBJECTED TO A KNOWN LOAD DISTURBANCE*

By H. E. Gollwitzer‡

16759 A

ABSTRACT

An open-loop optimal control problem is considered for plants that can be represented by linear recurrence equations. It is assumed that the control is bounded and that a known disturbance is present. Then the problem is to choose a control sequence that minimizes an error criterion based on a generalized distance function.

The problem is formulated in a manner such that linear programming techniques can be used to give the optimal control sequence. Estimates on the size of the resulting linear programming problem are presented.

A method is cited for determining an optimal control sequence as a result of varying a nominal disturbance provided the optimal control is known for the nominal case.

An example is presented to illustrate the techniques involved.

AUTHOR

INTRODUCTION

An open loop optimal control problem is considered for plants that can be represented by linear recurrence equations of the form:

---

* Prepared under contract NASw-563 for the NASA

‡ Associate Research Scientist, Minneapolis-Honeywell Reg. Co., Minneapolis, Minnesota

$$x(r+1) = A(r)\ x(r) + B(r)\ u(r) + g(r) \tag{1}$$

where $x(0) = x_0$; $r = 0,1,\ldots,\ell-1$; $x(r)$ is an n-vector; $g(r)$ a known bounded n-vector; and $u(r)$ an m-vector whose components are bounded in absolute value by one. $A(r)$ and $B(r)$ are bounded matrices with $\det(A(r)) \neq 0$.

The object is to choose a control sequence $\{u(r)\}$ that satisfy the constraints and minimize

$$C(u) = \max_{1 \le i \le n}\ \max_{1 \le r \le \ell} |x_i(r;\ u,\ g)| \tag{2}$$

where $x_i(r;\ u,\ g)$ denotes the ith component of a solution to (1).

The representation of a solution to (1) allows the problem to be solved as one of linear programming and estimates on the size of the resulting linear program are presented.

A method using linear programming techniques is cited that gives new optimal controls as a function of changes in the function $g(r)$ if an optimal solution is known.

A solution for a particular example is presented to illustrate the techniques involved.

ANALYSIS

The statement of the problem is developed and is interpreted as one of linear programming. Estimates on the size of the resulting linear program are developed.

The variation of parameters form of a solution to (1) is given by

$$x(r;u,g) = \Phi(r)x_o + \sum_{j=1}^{r} \Phi(r)\, \Phi^{-1}(j)\, [B(j-1)u(j-1) + g(j-1)] \quad (3)$$

where $r = 1,2,\ldots, \ell$ and $\Phi(r) = A(r-1)\, A(r-2)\, \ldots\, A(0)$ (reference 1). Let z be a nonnegative scalar variable such that $|x_i(r;u,g)| \leq z$, $r = 1,2,\ldots, \ell$, $i = 1,\ldots,n$. Then an optimal sequence $\{u(r)\}$ that minimizes $C(u)$ is found by minimizing z subject to

$$\begin{aligned} |x_i(r;u,g| &\leq z \\ |u^k(j)| &\leq 1 \end{aligned} \quad (4)$$

for $r = 1,2,\ldots, \ell$, $i = 1,\ldots,n$, $j = 0,1,\ldots,\ell-1$, $k = 1,\ldots,m$. This is a consequence of the manner in which the non-negative scalar variable z was introduced.

Since each component for each stage of (3) defines a hyperplane with respect to $u^i(j-1)$ where $u^i(j-1)$ denotes the i^{th} component of the m-vector $u(j-1)$, the problem can be stated as an equivalent linear programming problem. Since $|a| \leq b$ implies $a \leq b$ and $a \geq -b$, the problem can be stated as the linear programming problem: minimize z subject to

$$\begin{aligned} x_i(r;u,g) &\leq z \\ x_i(r;u,g) &\geq -z \\ u^k(j) &\leq 1 \\ u^k(j) &\geq -1 \end{aligned} \quad (5)$$

for $r = 1,2,\ldots,\ell$, $j = 0,1,2,\ldots,\ell-1$, $k = 1,\ldots,m$, $i = 1,\ldots,n$ where $x_i(r;u,g)$ is given by (3). This problem can now be solved by standard techniques in linear programming (reference 2).

ESTIMATES ON THE SIZE OF THE LINEAR PROGRAM

Although the linear programming problem (5) can be solved by standard methods it is not in a standard form so that these methods can be directly applied. Let A be a pxq matrix, y a q-vector, c a q-vector and b a p-vector. A canonical maximum or minimum linear programming problem is that of finding $y_i \geq 0$, $i = 1,\ldots p$ that maximize or minimize cy subject to

$$Ay = b \tag{6}$$

where cy denotes the inner product of c and y.

In order that standard methods such as the simplex method can be used in solving a linear programming problem, the problem must be reduced to an equivalent canonical problem. The problem (5) can be reduced to an equivalent canonical problem in various ways. New non-negative variables can be introduced in the $2\ell(m+n)$ inequalities of (5) so that equality always holds and each of the $u^k(j)$ can be represented as the difference of two non-negative variables (reference 2). If this is done the problem will be in a canonical form and a technique such as the simplex methods can be directly applied. In the equivalent canonical problem of (5) let N_1 be the number of equations and N_2 be the number of non-negative variables. Then depending on the specific problem

$$2\ell\,(n+\tfrac{m}{2}) \le N_1 \le 2\ell\,(n+m)$$
$$\ell(2n+3m)+1 \le N_2 \le 2\ell\,(n+2m)+1 \qquad (7)$$

where ℓ, n, m are defined in (1).

A PERTURBATION METHOD

A method is cited that allows one to calculate a new optimal control as a function of specified variations in the function g(r) of (1) if an optimal solution using g(r) is known.

As stated before the problem (5) must be reduced to the equivalent canonical linear programming problem of minimizing cy subject to

$$Ay = b \qquad y_i \ge 0,\ i = 1,\ldots,p \qquad (8)$$

before solution methods can be applied. A direct examination of (5) will show that g(r) will be found in the vector b of (8). There are methods of determining changes in the optimal solution of (8) if the components of b are allowed to vary in a prescribed manner from a nominal choice. These methods go under the titles of sensitivity analysis and parametric linear programming (reference 3). These methods are advantageous because there is less work involved in using these methods than to calculate a new optimal control for each choice of g(r).

A SAMPLE PROBLEM

The system of recurrence equations considered is the one associated with the second order differential equation

$$\ddot{x}(t) = u(t) + g(t) \tag{9}$$

If $x \overset{\Delta}{=} x_1$ and $\dot{x} \overset{\Delta}{=} x_2$ the system of recurrence equations in vector notation is

$$\begin{bmatrix} x_1(r+1) \\ x_2(r+1) \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(r) \\ x_2(r) \end{bmatrix} + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} [u(r) + g(r)], \tag{10}$$

with $x(0) \overset{\Delta}{=} \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$ and $r = 0,1,\ldots$

T is a parameter which is analogous to a sampling interval if t is a time parameter. The solution to (10) is given by (reference 1)

$$\begin{bmatrix} x_1(r) \\ x_2(r) \end{bmatrix} = \begin{bmatrix} 1 & rT \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} + \sum_{j=1}^{r} \begin{bmatrix} (\frac{1}{2} + r - j)T^2 \\ T \end{bmatrix} [u(j-1)+g(j-1)] \tag{11}$$

The recurrence equation leads to the following problem: given

$$x(r+1) = A(r)x(r) + b[u(r)+g(r)],\ r = 1,2,\ldots,\ell-1, x(0) = 0;$$
$$|u(r)| \leq 1,\ g(r) \text{ specified } r = 0,1,\ldots,\ell-1, \text{ where} \tag{12}$$

$A(r) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix}$ then find $u(j)$, $j = 0,1,\ldots,\ell-1$ satisfying (12) that minimize $C(u)$ where

$$C(u) = \max_{1 \leq i \leq 2} \max_{1 \leq r \leq \ell} |x_i(r;u,g)| \tag{13}$$

and $x_i(r;u,g)$ denotes the ith component of a solution to (12).

Three cases will be considered and each will in effect be a different representation of (9) over the interval $0 \leq t \leq 3$. The disturbance function is taken to have the same profile in time for each case considered.

THE FIRST APPROXIMATION

Let $T = 1$, $\ell = 3$ and let $g(j)$ be specified as follows; $g(0) = 2$, $g(1) = -2$, $g(2) = 2$. Then from (11)

$$\begin{aligned}
x_1(1) &= \tfrac{1}{2}\, u(0) + 1 \\
x_2(1) &= u(0) + 2 \\
x_1(2) &= \tfrac{3}{2}\, u(0) + \tfrac{1}{2}\, u(1) + 2 \\
x_2(2) &= u(0) + u(1) \\
x_1(3) &= \tfrac{5}{2}\, u(0) + \tfrac{3}{2}\, u(1) + \tfrac{1}{2}\, u(2) + 3 \\
x_2(3) &= u(0) + u(1) + u(2) + 2
\end{aligned} \tag{14}$$

The optimum sequence, according to the methods developed, is found by minimizing z, $z \geq 0$ subject to

$$\begin{array}{lllll}
\frac{1}{2}u(0) & & & +1 & \leq z \\
\frac{1}{2}u(0) & & & +1 & \geq -z \\
u(0) & & & +2 & \leq z \\
u(0) & & & +2 & \geq z \\
\frac{3}{2}u(0) & +\frac{1}{2}u(1) & & +2 & \leq z \\
\frac{3}{2}u(0) & +\frac{1}{2}u(1) & & +2 & \geq -z \\
u(0) & +\ u(1) & & & \leq z \\
u(0) & +\ u(1) & & & \geq -z \\
\frac{5}{2}u(0) & +\frac{3}{2}u(1) & +\frac{1}{2}u(2) & +3 & \leq z \\
\frac{5}{2}u(0) & +\frac{3}{2}u(1) & +\frac{1}{2}u(2) & +3 & \geq z \\
u(0) & +\ u(1) & +\ u(2) & +2 & \leq z \\
u(0) & +\ u(1) & +\ u(2) & +2 & \geq -z \\
u(0) & & & & \leq 1 \\
u(0) & & & & \geq -z \\
 & u(1) & & & \leq 1 \\
 & u(1) & & & \geq -1 \\
 & & u(2) & & \leq 1 \\
 & & u(2) & & \geq -1
\end{array}$$

This problem is one of linear programming and can be solved by standard techniques. An optimum sequence of $u(j)$, $j = 0,1,2$ along with $x^1(r)$, $x^2(r)$ and $g(r)$ is illustrated in Fig. 1.

THE SECOND APPROXIMATION

Let $T = \frac{1}{2}$ and $\ell = 6$. The results for this case are found similarily and are illustrated in Figure 2.

THE THIRD APPROXIMATION

Let $T = \frac{1}{3}$ and $\ell = 9$. The results for this case are found similarily and are illustrated in Figure 3.

CONCLUSIONS

An open-loop optimal control problem for plants that can be represented by linear recurrence equations is solved using linear programming techniques. The example illustrates the techniques used. The resulting linear programming problem gets very large if the system is of high order and if a large number of stages are considered. This might ultimately restrict the use of digital computers in obtaining a solution.

REFERENCES

1. Lee, E. B., "Recurrence Equations and Control of Their Evolution", Journal of Math. Anal. and Appl. 1963. Also, Harvey and Lee, "The Theory of Time Optimal Control of Linear Systems", MH MPG Report 1829-TR 4.

2. Gale, David, "Theory of Linear Economic Models", McGraw-Hill Book Company, Inc., New York, 1960.

3. Garvin, Walter W., "Introduction to Linear Programming", McGraw-Hill Book Company, Inc., New York, 1960.

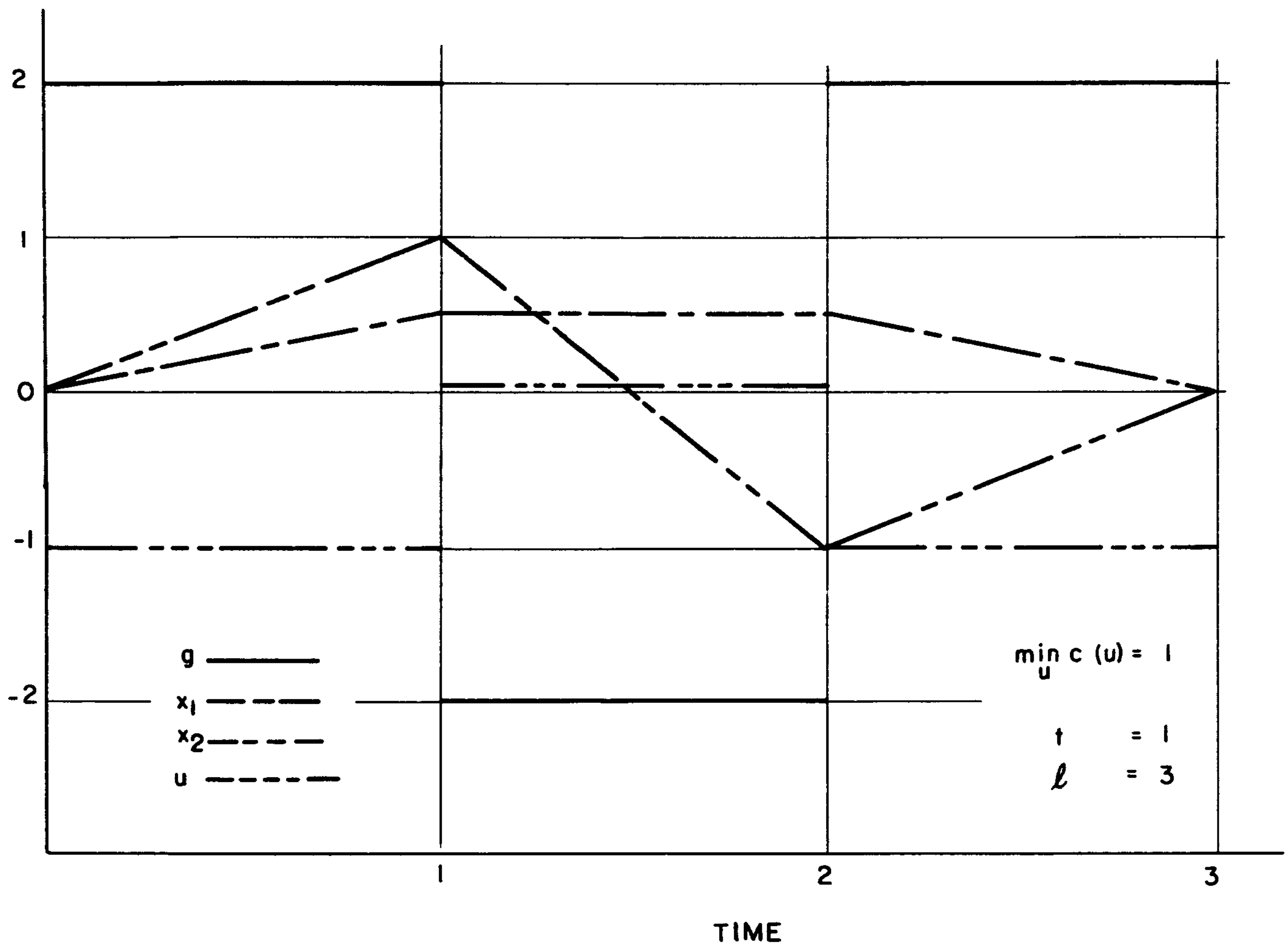

Figure 1. Results of the First Approximation

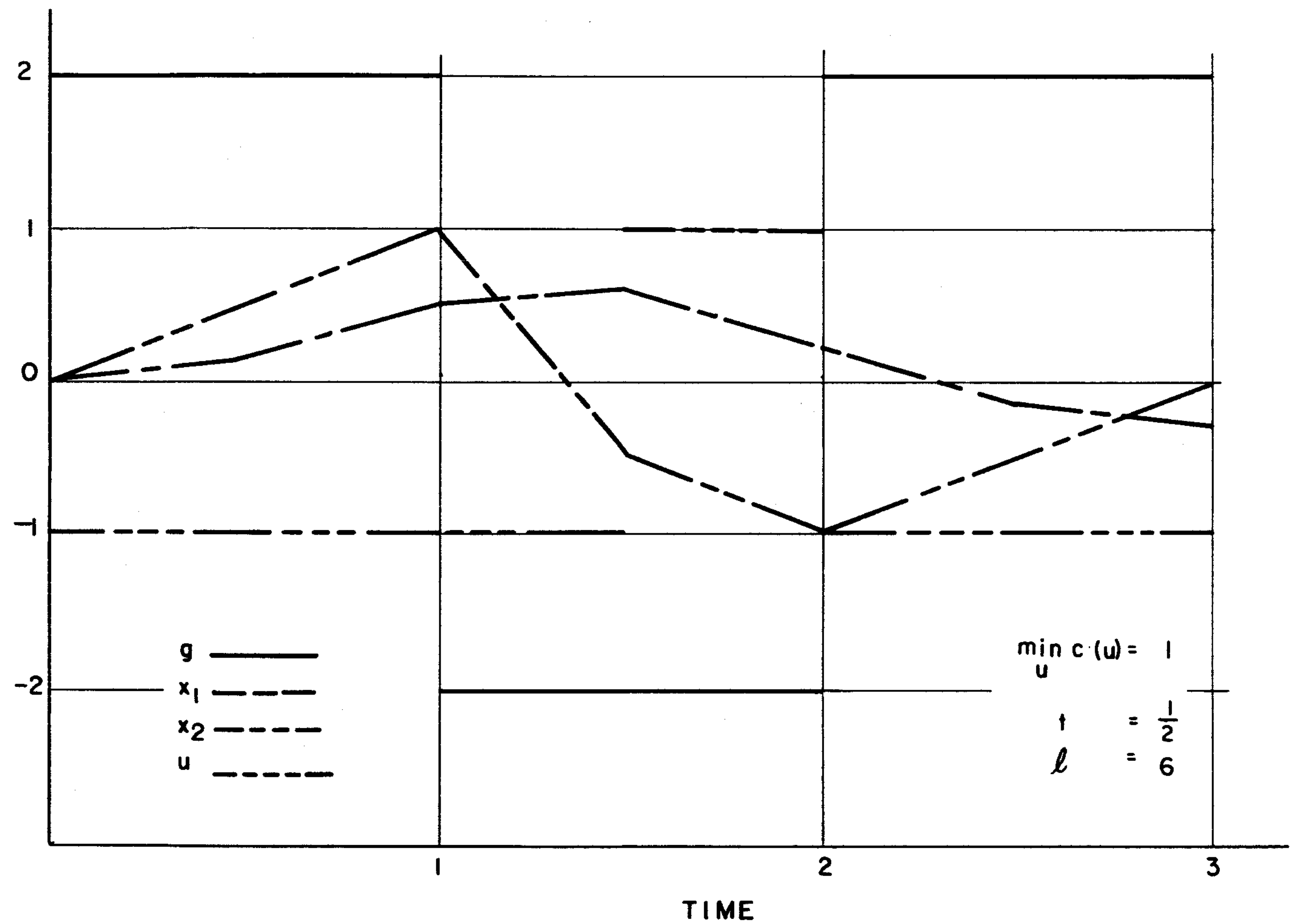

Figure 2. Results of the Second Approximation

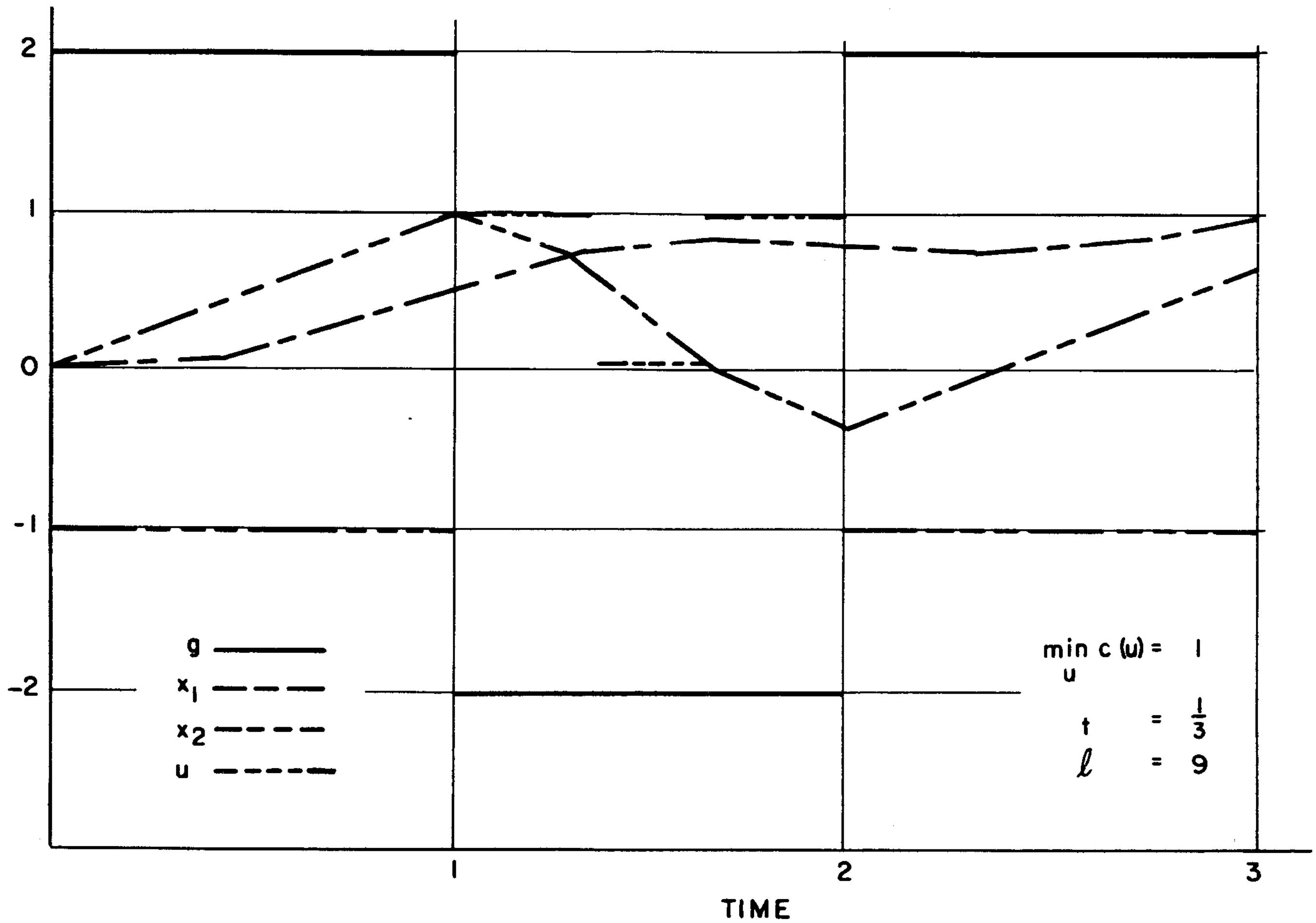

Figure 3. Results of the Third Approximation